$1/f$ noise in the two-body random ensemble

A. Relaño,[1] R. A. Molina,[2] and J. Retamosa[1]
[1]*Departamento de Física Atómica, Molecular y Nuclear, Universidad Complutense de Madrid, 28010 Madrid, Spain*
[2]*CEA/DSM, Service de Physique de l'Etat Condensé, 91191 Gif-sur-Yvette, France*
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We show that the spectral fluctuations of the two-body random ensemble exhibit $1/f$ noise. This result supports a recent conjecture stating that chaotic quantum systems are characterized by $1/f$ noise in their energy level fluctuations. After suitable individual averaging, we also study the distribution of the exponent α in the $1/f^{\alpha}$ noise for the individual members of the ensemble. Almost all the exponents lie inside a narrow interval around $\alpha=1$, suggesting that also individual members exhibit $1/f$ noise, provided they are individually unfolded.

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Our understanding of quantum chaos has greatly advanced during the last two decades. The pioneering work of Berry and Tabor [1] and Bohigas, Giannoni, and Schmit [2] showed that there exists a close relationship between the energy level fluctuation properties of a quantum system and the large-time-scale behavior of its classical analog. In their seminal paper, Bohigas *et al.* conjectured that the fluctuation properties of generic quantum systems, which in the classical limit are fully chaotic, coincide with those of random matrix theory (RMT). This conjecture is strongly supported by experimental data, many numerical calculations, and analytical work based on semiclassical arguments. Later the interest in these studies was renewed with the discovery that the spectral statistics of quantum-disordered systems is also well described by RMT. A review of later developments can be found in Refs. [3,4].

Thus, RMT plays a fundamental role in quantum chaos studies, though it was originally introduced by Wigner to describe the statistical properties of high-lying energy levels of quantum systems [5]. To describe spectral fluctuations, RMT assumes that physical Hamiltonians can be substituted by "reasonable" random ensembles of Hamiltonian matrices and introduces convenient statistics of their level spectra. RMT should provide the ensemble average of these statistics. Usually, these statistics are the nearest-neighbor spacing distribution [6] introduced to analyze the short-range correlations and the Dyson Δ_3 [6] statistic that allows one to study long range correlations.

Recently an approach, based on traditional methods of time series, has been proposed to analyze spectral fluctuations [7]. In this first work we showed that the classical random matrix ensembles (CRME's) exhibit $1/f$ noise in the fluctuations of the excitation energy. We also presented evidence that this is actually a universal property of quantum chaotic systems. The purpose of this Brief Report is to study whether the $1/f$ noise present in the spectral statistics of the CRME's is also present in the two-body random ensemble (TBRE). As we shall see below, this question is very pertinent if we want to apply this statistic to the study of the spectral fluctuations of real many-body systems.

The CRME's, usually called Gaussian and circular ensembles, were chosen due to their invariant properties under certain symmetry transformations [6]. For example, the Gaussian orthogonal ensemble (GOE) is invariant under orthogonal transformations and is applicable to systems invariant under time-reversal symmetry. However, the GOE represents systems with N-body interactions while normal systems in nature are supposed to be very well described by effective two-body interactions in the mean-field basis. The TBRE was introduced to tackle this problem and in this sense is more appropriate to study atomic nuclei, quantum dots, and other mesoscopic systems. *This ensemble is constructed from a GOE in the two-particle Hilbert space and then propagating it to the N-particle Hilbert space by using the direct product structure of this type of spaces* [for that reason this kind of ensembles is also called the embedded GOE (EGOE)] [8,9]. Given the single-particle states $|v_i\rangle$, $i\in 1,2,\ldots,M$, the two-body Hamiltonian is written as

$$H=\sum_{v_i<v_j,v_k<v_l}\langle v_k v_l|H|v_i v_j\rangle a^{\dagger}_{v_l}a^{\dagger}_{v_k}a_{v_i}a_{v_j}, \tag{1}$$

where $a^{\dagger}_{v_i}$ (a_{v_i}) creates (destroys) a fermion in the state $|v_i\rangle$. The two-body matrix elements $\langle v_k v_l|H|v_i v_j\rangle$ are properly antisymmetrized and are taken to be independent Gaussian random variables with

$$\overline{\langle v_k v_l|H|v_i v_j\rangle}=0,$$

$$\overline{|\langle v_k v_l|H|v_i v_j\rangle|^2}=\sigma^2(1+\delta^{c}_{(kl),(ij)}). \tag{2}$$

In this equation $\overline{\cdot}$ denotes ensemble average, σ is a constant, and δ^{c} is the Kronecker delta. Then, the Hamiltonian matrix in the N-particle space is defined in terms of these two-body matrix elements via the direct product structure. The only nonzero N-particle matrix elements are of three types

$$\langle v_1 v_2\cdots v_N|H|v_1 v_2\cdots v_N\rangle=\sum_{v_i<v_j\leq v_N}\langle v_i v_j|H|v_i v_j\rangle,$$

$$\langle v_p v_2\cdots v_N|H|v_1 v_2\cdots v_N\rangle=\sum_{v_i=v_2}^{v_N}\langle v_p v_i|H|v_1 v_i\rangle,$$

$$\langle v_p v_q \cdots v_N | H | v_1 v_2 \cdots v_N \rangle = \langle v_p v_q | H | v_1 v_2 \rangle, \tag{3}$$

or those obtained by permuting the single-particle states. All other matrix elements are zero.

Very few analytic results are known for the TBRE, contrary to the classical random matrix ensembles. A very important result is that the level density of the TBRE is Gaussian in the dilute limit, which corresponds to $(N,M)\to\infty$, $N/M\to 0$ [8,9], instead of the semicircular law for the GOE [6]. To perform a numerical analysis of the TBRE spectral statistics an important difficulty must be overcome: the TBRE is not ergodic [10–12]. In the present context ergodicity means that the statistical properties of individual ensemble members (and hence those of the physical Hamiltonian) should always coincide with the ensemble average. In order to transform the TBRE into an ergodic ensemble the spectrum of each member must be unfolded (see the unfolding description below) individually. In this way, the GOE statistics is recovered. For a recent review of the TBRE and more generally the EGOE see Ref. [13]

In order to establish whether $1/f$ noise is also present in the spectral fluctuations of the TBRE, we have studied four ensembles with different matrix sizes. We have treated $N=6$ "spinless" fermions in $M=11$, 12, 13, and 14 degenerated states, leading to Hilbert space dimensions $D=462$, 924, 1716, and 3003, respectively. The TBRE matrices were constructed using Eqs. (1), (2), and (3) with $\sigma=1$. There is no relevant energy scale in the model and the only parameter is the dimension of the Hilbert space. We have diagonalized 200 matrices in each case to obtain the ensemble average.

For each Hamiltonian matrix the level density $\rho(E)$ can be separated into a smooth part $\overline{\rho}(E)$, which defines the main trend of the level density, and a fluctuating part $\widetilde{\rho}(E)$. It is well known that level fluctuation amplitudes are modulated by $\overline{\rho}(E)$; therefore, to compare the statistical properties of different systems or different parts of the same spectrum, the main trend defined by $\overline{\rho}(E)$ must be removed. This procedure, called unfolding, consists in mapping the level energies E_i into new dimensionless levels ϵ_i:

$$E_i \longrightarrow \epsilon_i = \overline{\Sigma}(E_i), \quad i=1,2,\ldots,D, \tag{4}$$

where $\overline{\Sigma}(E)$ is a smooth approximation to the actual step function $\Sigma(E)$ that gives the true number of energy levels from the ground-state energy E_0 and up to energy E. This function is given by

$$\Sigma(E) = \int_{E_0}^{E} \rho(\eta) d\eta. \tag{5}$$

We have already commented on the important analytical result of French and Wong [8] who showed that in the TBRE the mean level density $\overline{\rho}(E)$ goes to Gaussian form in the dilute limit. However, for the dimensions of the matrices used in this work, the corrections to the Gaussian behavior are very important and different for each matrix. Since the use of an accurate unfolding procedure is essential to avoid misleading results for the long-range spectral correlations [14], we have selected another method. Recently, the problems related to the unfolding procedure in the TBRE have been discussed deeply in Refs. [15,16]. After some tests we have chosen polynomials up to grade 5 to fit the accumulated level density $\Sigma(E)$; higher grades produce spurious long-range correlations. Finally, we have thrown 5% of the eigenvalues in the two spectrum edges.

In the approach of Ref. [7], the analogy of the energy spectrum with a time series is established in terms of the δ_q statistic. Using the unfolded energies it is defined as [6]

$$\delta_q = \sum_{i=1}^{q} (s_i - \langle s \rangle) = \epsilon_{q+1} - \epsilon_1 - q, \tag{6}$$

where s_i are the next-neighbor level spacings, $s_i=\epsilon_{i+1}-\epsilon_i$, with spectral average value $\langle s\rangle=1$. Note that δ_q represents the deviation of the excitation energy of the $(q+1)$th unfolded level from its mean value. Moreover, it is closely related to the level density fluctuations. Indeed, we can write

$$\delta_q = \overline{\Sigma}(E_{q+1}) - \Sigma(E_{q+1}) = -\widetilde{\Sigma}(E_{q+1}), \tag{7}$$

if we appropriately shift the ground-state energy; thus, it represents the accumulated level density fluctuations at $E=E_{q+1}$.

We will profit from the formal similarity of the δ_q function with a time series to analyze its properties with numerical techniques, normally used in the domain of complex systems. The most simple procedure is to study the scaling properties of its power spectrum $S(k)$. The latter is defined in terms of the discrete Fourier transform

$$\hat{\delta}_k = \frac{1}{d}\sum_{q=1}^{d-1} \delta_q \exp\left(\frac{2\pi i k q}{d}\right) \tag{8}$$

in the usual way as

$$S(k) = |\hat{\delta}_k|^2, \tag{9}$$

where $d \leqslant D$ is the total number of unfolded levels considered. In the present work $d \simeq 0.9D$. We will say that the spectral fluctuations of a Hamiltonian ensemble exhibit $1/f$ noise if the ensemble-averaged power spectrum of δ_q follows a power law of type

$$\overline{S(k)} \propto \frac{1}{k^{\alpha}}, \tag{10}$$

with $\alpha \simeq 1$. For a single Hamiltonian it is not clear whether its bare power spectrum or some kind of average follows the previous power law. We shall explore three different possibilities below.

The results obtained for the ensemble-averaged power spectra are shown in Fig. 1 using a log-log scale. It is clearly seen that the calculated points spread along straight lines. The line slopes,—i.e., the power spectrum exponents—are obtained by means of a least-squares fit and their values are $\alpha=1.09\pm0.04$, $\alpha=1.08\pm0.01$, $\alpha=1.07\pm0.01$, and $\alpha=1.07\pm0.01$ for $(N,M)=(6,11)$, (6,12), (6,13), and (6,14), respectively. The exponents are very close to 1, confirming that there is $1/f$ noise in the TBRE. Thus, we obtain a new and powerful check of the conjecture that links the spectral statistics of the TBRE with that of the GOE.

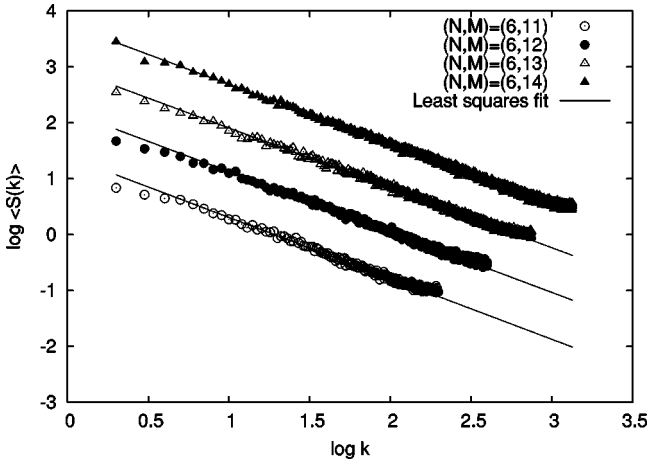

FIG. 1. Ensemble-averaged power spectra of four TBRE's with different dimensions. The best $1/f^{\alpha}$ fit is also shown. The curves have been displaced vertically to avoid overlapping between them.

Moreover, in order to study to what extent the δ_q statistic is also meaningful for individual spectra, we have randomly selected a member pertaining to the TBRE ensemble with $(N,M)=(6,14)$. The upper plot of Fig. 2 shows the power spectrum of the δ_q function for this member. Although this result suggests the existence of a power law, the calculated points are widely spread around the mean behavior, and therefore other different curves can be used to fit the data points. Performing a least-squares fit to a straight line we obtain a value $\alpha=1.10\pm0.07$, but the error seems unreliable. Following Mandelbrot, the problem arises because of the double-logarithmic plot: *spectral components must never be plotted raw, only after suitable averaging* [17]. One of the best procedures to perform this average consists in dividing

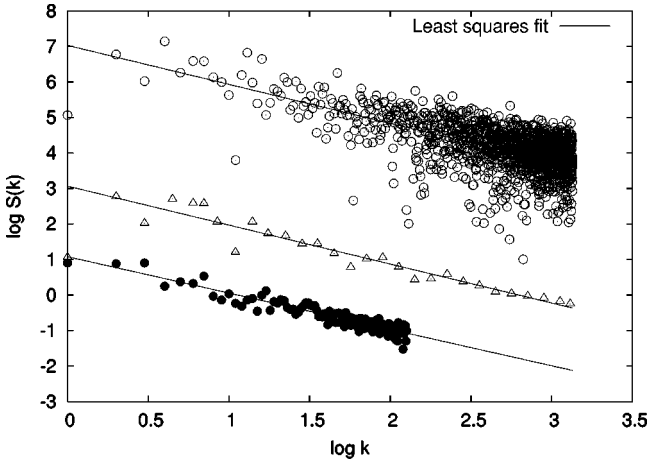

FIG. 2. Three examples of power spectra for an individual Hamiltonian selected from the $(N,M)=(6,14)$ TBRE. The curves have also been displaced vertically to avoid overlapping. Open circles represent the raw power spectrum, open triangles the average values of the power spectrum obtained by binning the log (frequency) scale, and solid circles the running-averaged power spectrum. The solid line represents the best fit in each example. The power spectrum exponents are $\alpha=1.10\pm0.07$, $\alpha=1.10\pm0.06$, and $\alpha=1.02\pm0.05$, respectively.

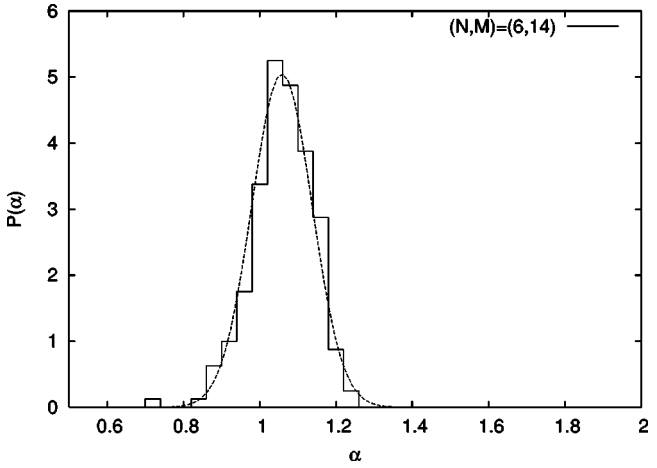

FIG. 3. Histogram of the distribution of α for the $(N,M)=(6,14)$ TBRE.

the high-frequency portion of the logarithmic frequency axis into equal bins and averaging the power spectrum components in each bin. The result of this procedure is shown in the second plot of Fig. 2: the averaged data points are no more widely spread, but all of them fall near the mean behavior. If we perform a least-squares fit to this new set of data points we obtain $\alpha=1.10\pm0.06$; therefore, the fuzzy behavior is confirmed to be related just to the double-logarithmic plot. An alternative averaging procedure is to calculate a *running or spectral average* $\langle S(k)\rangle$. Since in this case the dimension is large enough, we can divide the whole spectrum into 10 different sets of 256 consecutive levels. Then, the δ_q power spectrum is calculated for each level set and in order to reduce fluctuations and clarify the main trend a running average is performed using these sets. The bottom plot of Fig. 2 displays the result of this calculation using open circles. A least-squares fit leads to the exponent $\alpha=1.02\pm0.05$, which is very similar to that obtained by means of an ensemble average. This result is compatible with the ergodic properties of the TBRE once the spectra are individually unfolded. Note that the different lengths of the three power spectra shown in this figure are due to the fact that in the first and second cases we are actually using the whole sequence, but in the third one we use sequences of 256 consecutive levels.

To make this discussion more quantitative, we have calculated the exponent α for the 200 matrices of the ensemble with $(N,M)=(6,14)$ using the binning method previously described. The average value is $\alpha=1.06$ and the width of the distribution is $\sigma_{\alpha}=0.08$. Figure 3 shows a histogram of the distribution together with a Gaussian defined by the previous parameters that seems to fit the data very well. Although this result has been obtained for a particular sample of a particular ensemble $[(N,M)=(6,14)]$, it suggests that individual members also are characterized by $1/f$ noise.

We have confirmed that the spectral fluctuations of the TBRE exhibit $1/f$ noise. This behavior supports the previously stated conjecture that chaotic quantum systems are characterized by $1/f$ noise in their energy level fluctuations. We have also shown that individual members have $1/f$ noise in their excitation energy fluctuations provided they are indi-

vidually unfolded and the power spectrum of the δ_q function is appropriately averaged. Actually, the distribution of the α exponent in the $1/f^{\alpha}$ law is a Gaussian centered near $\alpha=1$ with a quite small width. Therefore, the spectral fluctuations of atomic nuclei, quantum dots, and mesoscopic systems can be studied by means of the scaling properties of the power spectrum of the δ_q function. The advantages of this new statistic are perfectly used in some recent works about the nuclear masses [18,19], where the $1/f$ noise of different series of fluctuations in the nuclear masses along the nuclear chart was explored. Depending on the definition of the fluctuations, different exponents in the power law were found. $1/f$ noise was shown to be a powerful tool to investigate spectral correlations in this kind of experimental data.

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